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APPROXIMATION OF YIELD CONDITION FOR THE HYSTERETIC BEHAVIOR OF A BAR UNDER REPEATED AXIAL LOADING

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Abstract—This paper is concerned with the hysteretic behavior of a prismatic bar subjected to repeated axial loading. Elastic-perfectly plastic behavior is assumed in the analysis under the combined action of axial force and bending moment. The fully plastic states in pure bending and in pure tension and compression are bilinearly interpolated to serve as the yield condition; linearly interacted regimes are combined with interactionless regimes of pure bending to form the yield hexagon, which is a modification of a previously assumed yield quadrangle. Basic equations are derived through the analysis, expressed in a simple analytic closed form, and are able to determine the load-deformation relationship of the bar for any specified history of axial loading.

As a result of the analysis it follows that due to the load cycle of tension and compression an initially straight and plastically bent bar undergoes a plastic extension upon the recovery of the straight configuration. This is the balanced axial deformation at a yield hinge between extension and contraction, taking place during the cycle of hinge rotation. Another characteristic feature found is that cyclic alternate displacement loading with large amplitude leads to a steady state after some repeated deterioration of processes. Comparison is made of the load-axial displacement relation between the analytical and experimental results to show reasonable agreement, with discussions extended to the appropriateness of the form of the yield condition.

1. INTRODUCTION

Axially loaded members play an important role in such structures as trusses and braced frames. Clarification of the performance of these structures under repeated loading requires the knowledge of the hysteretic load-deformation characteristics of axially loaded members. Recently in Italy and Japan much effort has been directed to the formulation of the elastic–plastic axial load-displacement relation of steel braces, in relation to the response of tall braced frames to earthquake excitation; the static relationship is useful for the determination of dynamic behavior, since braces have relatively small mass. It is worthy of note that plastic action in axially loaded members ordinarily takes precedence over flexurally loaded members because of the predominance in stiffness of the former over the latter. Some results of numerical analyses are reported in the form of charts[1–7], and a successive approach is proposed on the basis of simple plastic theory established for beams and unbraced frames, which does not take account of plastic axial deformation[8, 9].

The author has formulated a general elastic–plastic solution for the hysteretic behavior of a prismatic bar subject to repeatedly applied loads in the axial direction[10]. The assumption of perfect plasticity together with the one-dimensional idealization of the bar has led to basic equations which have been shown to be adequate for the behavior to be determined for any history of axial loading in the range of small deformation. This theory describes how a bar, after deforming plastically due to instability during compression, recovers in a subsequent tension, and describes how the bar as a structural member gets loosened by plastic elongation, reducing the overall stiffness of the structure. While the usefulness of the theory as a first order approximation has been confirmed experimentally[11, 12], it contains a significant defect. The analysis has been based on the assumption that the fully plastic states in pure bending and in purely axial loading can be linearly interpolated to serve as the yield condition for the interaction of bending moment and axial force. The piecewise linearity in the yield condition is the idealization for an I-section with thin web and flanges, and is rather a crude approximation of commercially available cross sections. As a consequence, the solution has had a merit in simplicity but a demerit in the prediction that a plastically deformed bar completely restores its full strength and initial straightness upon the reversion of the axial displacement, as against the actual observation that a plastically deformed bar hardly becomes straight again through mere extension. A nonlinear yield condition replacing the piecewise-linear one improves the prediction on this point to a certain extent, and an attempt to use a parabolic yield condition has resulted in better agreement with the experimental behavior of a rectangular prism [13]. Then, however, the basic equations have not been able to be expressed in a finite number of analytic functions. The objective of the paper presented herein is to obviate the two shortcomings found in the use of the piecewise-linear and parabolic yield conditions, and to develop an analysis which still leads to a closed-form analytic solution. This is accomplished by modifying the form of the yield condition. The analytical results are then compared with the experimental, and discussions extend to the pertinence of the modification.

2. ASSUMPTIONS

An initially straight bar of uniform material and cross section is subjected to a repeatedly applied load, which is a pair of equal and opposite forces acting centroidally at the ends of the bar with slowly varying intensity. The effective length L is taken so that the bar can be regarded as being simply supported at its ends. The main interest lies in the hysteretic relationship between the load N, positive for tension, and the relative displacement Δ, positive for separation, of the bar ends in the direction of loading. In this paper there is the restriction that deformation may be finite but small enough so that change in the length of the bar is negligible when compared with the original and so that the square of the slope of the deflection curve can be neglected in comparison with unity. It is assumed that the cross section has axes of symmetry and that the bar deflects only in the plane of symmetry without twist, having cross-sectional area A and moment of inertia I. The bar is idealized as a one-dimensional continuum, having the property of linear elasticity with Young's modulus E followed by perfect plasticity under combined action of the axial force N and bending moment M.

The yield condition for the M–N interaction is approximated by bilinearly interpolating the fully plastic state $|M| = M_0$ in pure bending and the fully plastic state $|N| = N_0$ in pure tension and compression. This is represented in the dimensionless stress-resultant plane (M/M_0, N/N_0) to form the yield hexagon, as shown by solid lines in Fig. 1. Yielding is stipulated by linking interactionless regimes of pure bending with linearly interacted regimes, having a corner at $(1, 1-c)$ in the first quadrant, and having double symmetry with respect to the co-ordinate axes. This form of the yield condition is selected as the simplest with a single parameter c among bilinear representations in one quadrant, while still furnishing a reasonable prediction for the overall hysteretic behavior of an axially loaded bar; c is to be chosen properly between zero and unity, according to the shape of the cross section and the dominant range of the load intensity. The choice $c = 1$ would be identical with the previously assumed piecewise-linear yield condition which is represented by the square as drawn in dotted-and-dashed lines in Fig. 1. If it is specified that the same area of elastic domain be enclosed by the hexagon of the modified piecewise-linear yield condition and by the pair of parabolas drawn in dotted lines which correspond to the fully plastic condition for a rectangular cross section, then it follows that $c = 2/3$. It is worth noting that the modified piecewise linearity is equivalent to assuming the cross section to consist of three concentrated areas, two as flanges and the other as the center web.

It is also assumed that a compressed straight bar buckles when the compression reaches

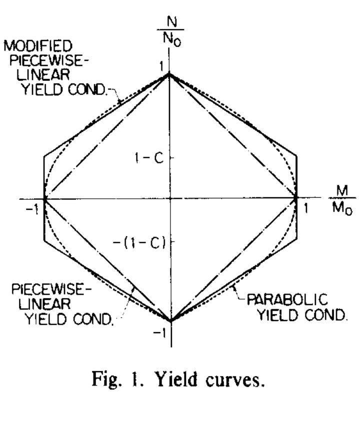

Fig. 1. Yield curves.

either the Euler load $N_E = \pi^2 EI/L^2$ or the crush load N_0. Investigation is focused on bars of moderate slenderness such that N_E and N_0 are of the same order of magnitude. The effects of shear are completely neglected.

3. FORMULATION

While taking note of the fact that the history of the axial displacement has to be specified in order for the behavior of the bar to be determined, it is convenient to express the displacement in terms of the load. Some of the basic relations established previously remain valid[10]. The dimensionless displacement δ, the ratio of the relative axial displacement to the yield point displacement, is composed of four components, viz.,

$$\delta \equiv (EA/N_0L)\Delta = \delta^e + \delta^g + \delta^p + \delta^t,$$

where δ^e is due to elastic axial deformation, $\delta^g \leqq 0$ to change in geometry associated with transverse deflection, δ^p to plastic axial deformation at a yield hinge, and $\delta^t \geqq 0$ to plastic elongation in a straight configuration. Among these it is clear that δ^e is equal to the dimensionless load $n \equiv N/N_0$, and that δ^t can increase when n equals unity and remains constant otherwise. The component δ^g is caused by the difference in length between the arc and chord of the deflection curve, two halves of which are symmetric and determined from the equilibrium and elasticity of the bar. Each half is expressible in terms of a hyperbolic function; see Fig. 2. At the middle the deflection $V \equiv (M_0/N_0)v \geqq 0$ is related to the slope angle $\Theta \equiv (2M_0/N_0L)\theta \geqq 0$ by

$$\theta = (\nu/\tanh \nu)v. \tag{1}$$

Integration of the square of the slope along the bar axis provides the component

$$\delta^g = -\alpha n_E \left(\frac{\theta}{\pi \cosh \nu}\right)^2 \left[\frac{\sinh(2\nu)}{2\nu} + 1\right], \tag{2}$$

where $\alpha \equiv (A/I)(M_0/N_0)^2$ is a cross-sectional constant, $n_E \equiv N_E/N_0$ is a slenderness constant, and $\nu \equiv (NL^2/4EI)^{1/2} = (\pi/2)(n/n_E)^{1/2}$ is a dimensionless axial force. When the load is negative, ν is imaginary and in order to express the results in terms of real functions, it suffices to replace ν by its modulus and to replace the hyperbolic functions by the corresponding trigonometric functions. The above relations are unchanged by the modification of the yield condition.

Plastic action can take place with a yield hinge at the middle of the bar when the deflection attains such a value that the yield condition is satisfied. In order to relate this deflection to the load, it is first noted from the equilibrium relation

$$m + nv = 0 \tag{3}$$

that the bending moment $M \equiv mM_0$ at the middle has the opposite sign to the load when the positive moment is defined to be compatible with an increase in the hinge rotation 2Θ as shown in Fig. 2. It follows that the state of stress at the middle cross section has a stress point somewhere in the second or fourth quadrant in the stress plane of Fig. 1, and for the relevant regimes the yield condition reads

$$|m| = 1 \quad \text{for} \quad |n| \leqq 1 - c, \tag{4a}$$

$$|n - cm| = 1 \quad \text{for} \quad 1 - c \leqq |n| \leqq 1. \tag{4b}$$

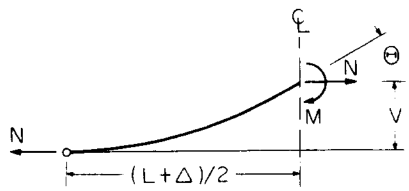

Fig. 2. Notation.

Combination of eqns (3) and (4) gives

$$v = \begin{cases} 1/|n| & \text{for} \quad |n| \leqq 1 - c \\ (1/c)(1/|n| - 1) & \text{for} \quad 1 - c \leqq |n| \leqq 1. \end{cases} \tag{5a,b}$$

As long as eqn (5) is kept satisfied, there can be a change in the hinge rotation at the middle. It is accompanied by a change in the plastic axial deformation, their relation being determined by the normality of the flow rule associated with eqn (4). It follows for the increments that

$$\mathrm{d}\delta^p = \begin{cases} 0 & \text{for} \quad |n| < 1 - c \\ -(2/\pi)^2(\alpha n_E/c)\,\mathrm{d}\theta & \text{for} \quad 1 - c < |n| < 1. \end{cases} \tag{6a, b}$$

The hinge rotation to be substituted in eqns (2) and (6) is related to the load by combining eqns (1) and (5) for a plastic process.

When the deflection is smaller than is given by eqn (5), then the bar behaves in an elastic manner and

$$\mathrm{d}\delta^p = \mathrm{d}\theta = 0. \tag{7}$$

If v is to be determined for an elastic process, eqn (1) may be used with θ kept constant and specified by the preceding plastic action. It is worthy of note that because of the incremental relation of eqn (6) the hereditary nature of plasticity is comprised not only in δ^t and θ but also in δ^p, appearing as an integral constant, which has not been the case with the piecewise-linear yield condition assumed previously[10].

4. ILLUSTRATION

Figure 3 serves to illustrate the behavioral characteristics of a bar under a varying axial load. In (a)–(d) of Fig. 3, δ, v, θ and m are taken, respectively, for the abscissae as against the common ordinate of n. The variation in the states of deformation and stress is indicated by orderly numbering in circles. It is assumed in this illustration that $1 - c < n_E < 1$. After being subjected to plastic yielding in tension, the bar is compressed until it buckles at $n = -n_E$ as in 0 → 3. A yield hinge is formed at the middle of the bar in 4 → 6, with the stress point moving first along an interacted regime and transferring at 5 to an interactionless regime of the yield hexagon as shown in Fig. 3(d). During the process 4 → 5 the yielding causes plastic flow of contraction in the axial direction besides a relative rotation across the yield hinge, whereas no change takes place in the axial plastic deformation in 5 → 6. The process 6 → 7 is characterized by elastic recovery, and is followed by hinge action with a tensile force in 7 → 9, until the bar restores straightness, $v = \theta = 0$, with $n = 1$ at 9. The plastic axial deformation developed at the yield hinge in 8 → 9 is an extension, and the balanced hinge extension remaining at 9 is given through eqns (6) and (7) from

$$\delta_9{}^p = -(2/\pi)^2(\alpha n_E/c)[(\theta_9 - \theta_8) + (\theta_5 - \theta_4)], \tag{8}$$

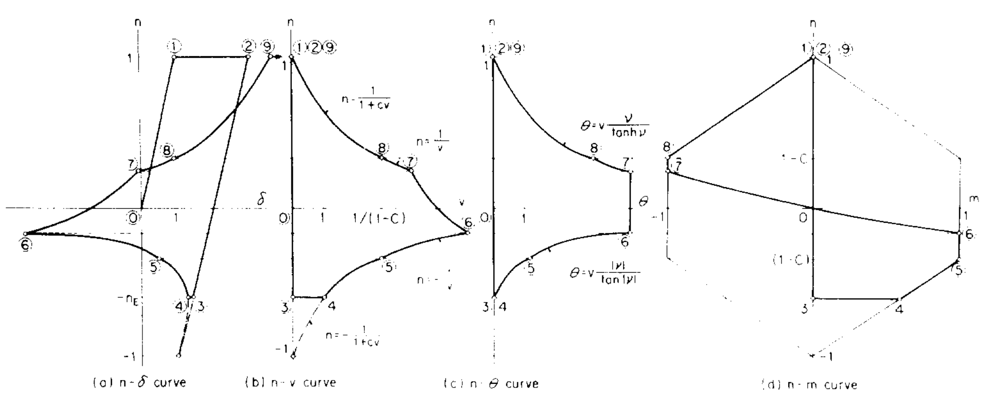

Fig. 3. Behavioral diagrams of an axially loaded bar.

where subscripts refer to the states defined above. Clearly $\theta_9 = \theta_4 = 0$. With $n_8 = -n_5 = 1 - c$, and with eqn (5), it is found that $v_8 = v_5 = 1/(1-c)$, and hence eqns (1) and (8) are combined to give

$$\delta_9^p = \frac{2}{\pi}\frac{\alpha}{c}\left(\frac{n_E}{1-c}\right)^{1/2}\left\{1/\tanh\left[\frac{\pi}{2}\left(\frac{1-c}{n_E}\right)^{1/2}\right] - 1/\tan\left[\frac{\pi}{2}\left(\frac{1-c}{n_E}\right)^{1/2}\right]\right\} > 0. \tag{9}$$

The fact $\delta_9^p > 0$ follows from the inequality $\theta_8 > \theta_5$, which is visualized by noting that there is a change in the hinge rotation without a change in the plastic axial deformation between the states ⑤ and ⑧, the middle deflection returning to the identical value, and by noting that the actual deflection curve is hyperbolic-sine for tension and is sine for compression in each half with the co-ordinate origin at the end, as shown in Fig. 4. It is thus seen that the yield hinge undergoes extension through the rotation cycle caused by the load cycling from and to $n = 1$. Therefore, the n–δ curve of Fig. 3(a) does not close in ②→⑨; since the changes in δ^e, δ^g and δ^t between ② and ⑨ all vanish, the difference, $\delta_9 - \delta_2$, equals δ_9^p of eqn (9). For a rectangular cross section, $\alpha = 3/4$, and specification $c = 2/3$, $n_E = 3/5$, for example, leads to $\delta_9 - \delta_2 = 0.76$. It is reminded that in the previous case of the piecewise-linear yield condition the accumulation phenomenon in hinge extension has not occurred, but the n–δ curve has closed at the attainment of $n = 1$, because of the complete proportionality of the hinge rotation and plastic axial deformation[10].

Let us examine a case of cyclic displacement loading. It is again assumed that $1 - c < n_E < 1$. The loading is started with contraction and is repeated alternatively with the same amplitude δ_A in the separation and approach. If δ_A is sufficiently large, the hysteretic n–δ relation traces such loops as shown in Fig. 5. The behavior of the bar deteriorates with the number of repetition of loading cycles until the load magnitude is reduced into the range $|n| \leqq 1 - c$. Henceforth the loop becomes stable as drawn in thick lines, and the behavior reaches a steady state. The previous piecewise linearity in the yield condition has indicated for a steady state to be attained before the end of the first cycle[10], and the use of the parabolic yield condition never leads to a steady state, predicting everlasting deterioration[13].

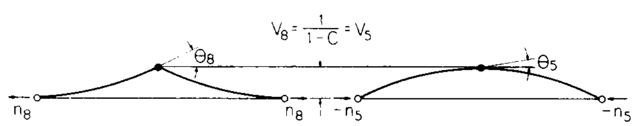

Fig. 4. Dimensionless deflection curves.

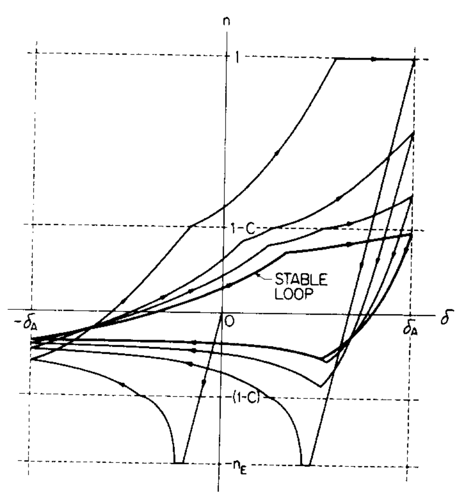

Fig. 5. Cyclic alternate displacement loading.

5. COMPARISON AND CONCLUSIONS

Comparison is made of the n–δ relation between the theoretical and experimental results in Fig. 6. The solid lines are drawn for the modified piecewise-linear yield condition with $c = 2/3$, dotted lines for the parabolic yield condition[13], dotted-and-dashed lines for the piecewise-

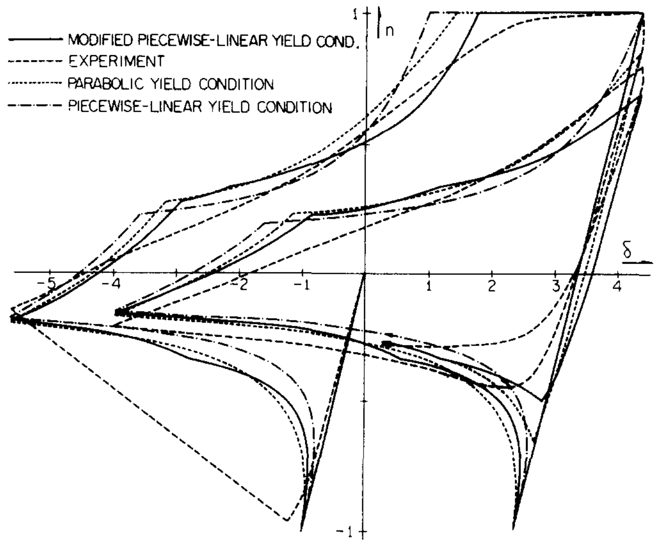

Fig. 6. Comparison between theories and experiment.

linear yield condition[10], and dashed lines for experiment[14, 15]. The tests were carried out with mild steel specimens of square cross section (15 × 15 cm). The particular specimen cited had the slenderness of $n_E = 1.13$. The loading program was controlled by the axial displacement, with regard to which discretion is exercised in comparing with theoretical prediction. Starting with contraction, the loading was not successfully regulated in the early post-buckling region because of the rapid variation in the load and deformation. This is seen to cause a marked discrepancy with the theories in this region. For another discrepancy observed near the second buckling region the theories may be responsible; all the theories introduced herein predict the complete recovery of straightness upon the attainment of $n = 1$, material hardening in the plastic range being ignored, whereas the actual bar hardly becomes straight through mere extension after being plastically deformed into a crooked configuration. Except for these regions and for other minor discrepancies occurring due to the theoretical negligence of partial yielding in cross sections, general agreement is seen between the theoretical and experimental results. Difference in the theoretical prediction due to the various forms of the yield condition is rather remarkable. It is seen that the parabolic or modified piecewise-linear yield condition with $c = 2/3$ better fits the experimental behavior of the mild steel square prism than the piecewise-linear yield condition, which corresponds to a state of stress lying somewhere between the initial yield and full plasticity for many practical cross sections. The discrepancy between the modified piecewise-linear and parabolic yield conditions is not significant in this figure, but an advantage in the use of the former lies in the simplicity of analysis. Another advantage is concerned with the experimental observation that steel bars tend visibly to arrive at a steady state, which can be reached in the prediction by the former but not by the latter, within the first few cycles when subjected to cyclic displacement loading, unless local instabilities or twisting occurs[15]. Within the framework of this series of theoretical investigation, therefore, it seems advisable to use the modified piecewise-linear yield condition with a proper choice of the parameter c depending on the cross-sectional shape and the dominant loading range.

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